St George Girls High School

Year 12

Mid-HSC Course Examination

2010

Mathematics Extension 1

General Instructions

- Working time – 1½ hours
- Reading time – 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question in a **new booklet**.

Total marks – 72

- Attempt Questions 1 – 6
- All questions are of equal value

Question 1 – (12 marks) – (Start a new booklet) **Marks**

a) Find the value of k 2

$$\log_e 5 - \tfrac{1}{2}\log_e 16 = \log_e k$$

b) (i) Express $\dfrac{2\pi}{9}$ radians in degrees. 1

(ii) Find the exact value of 2

$$\cos\left(\frac{-11\pi}{6}\right) + \sin\left(\frac{2\pi}{3}\right)$$

c) Find the derivative of $y = \log_e\left(\frac{x+1}{\sqrt{x}}\right)$ expressing your answer as a single algebraic fraction. 3

d) Find the equation of the normal to the curve $y = \log_e(x^2+2)$ at the point where $x = 1$ 4

Question 2 – (12 marks) – (Start a new booklet) **Marks**

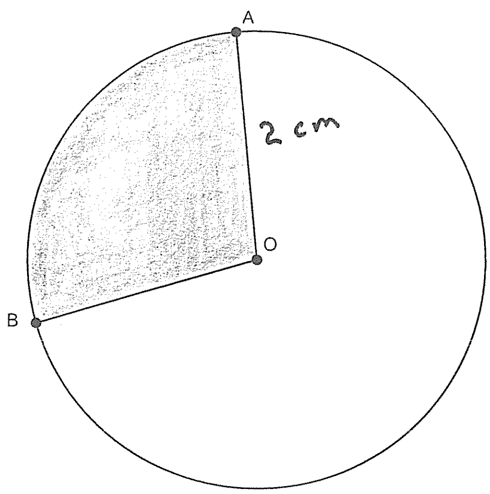
a) Find $\lim_{x \to 0} \dfrac{\sin\left(\frac{2x}{5}\right)}{3x}$ 2

b) (i) Show that $\cos 4x = 2\cos^2 2x - 1$ 1

(ii) Hence find $\int \cos^2 2x \; dx$ 2

c) Evaluate $\int_0^{\log_e 2} \dfrac{e^{3x}}{1+e^{3x}} \, dx$ 3

d)



The area of the minor sector AOB is 5 cm^2.

(i) Find the size of $\angle AOB$ 2

(ii) Hence, find the length of the major arc AB (correct to 1 decimal place) 2

Question 3 – (12 marks) – (Start a new booklet) **Marks**

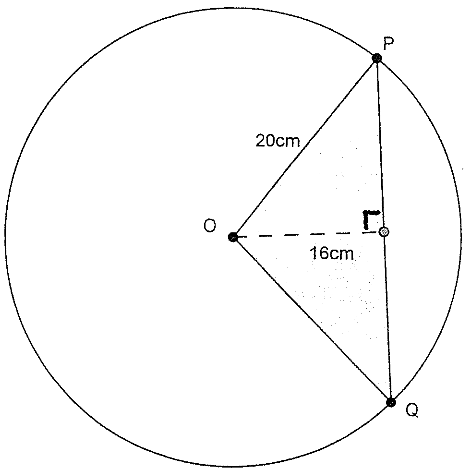
a) (i) Show that the line $y = 2 + 2x$ crosses the curve $y = \ln(3x + 4)$ at the point $(-1, 0)$ 1

(ii) Find the acute angle between $y = 2 + 2x$ and $y = \ln(3x + 4)$ at the point $(-1, 0)$. [answer to nearest minute] 3

b) Evaluate 2

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^2 x \, dx$$

c) Calculate the area of the minor segment cut off by the chord PQ, given the perpendicular distance from the centre O to PQ is 16cm. 3



d) Draw a neat sketch of the curve $y = \ln(3 - x)$ clearly indicating any x, y intercepts and asymptote(s). 3

Question 4 – **(12 marks)** – (Start a new booklet) **Marks**

a) (i) Find $\frac{d}{dx}\left[\frac{\log_e x}{x^2}\right]$ 2

(ii) Hence, (or otherwise) find the equation of the tangent to the curve
$y = \frac{\log_e x}{x^2}$, at $x = e$ 2

b) The area between the curves $y = \cos x$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{4}$ is rotated about the x-axis. 3

Calculate the volume of the solid formed. [give in exact form]

c) For the curve $y = e^{\sin x} - 1$ in the domain $0 \leq x \leq 2\pi$ find:

(i) the x-intercepts 2

(ii) the stationary points and determine their nature. 3

Question 5 – (12 marks) – (Start a new booklet) **Marks**

a) (i) Sketch the graph of $y = \sin 2x$ in the domain $0 \leq x \leq 2\pi$ [at least $\frac{1}{3}$ page] **2**

(ii) On the same graph, sketch the line required to determine the number of solutions for the equation $\frac{x}{2} + \sin 2x = 1$ **2**

(iii) How many solutions are there? **1**

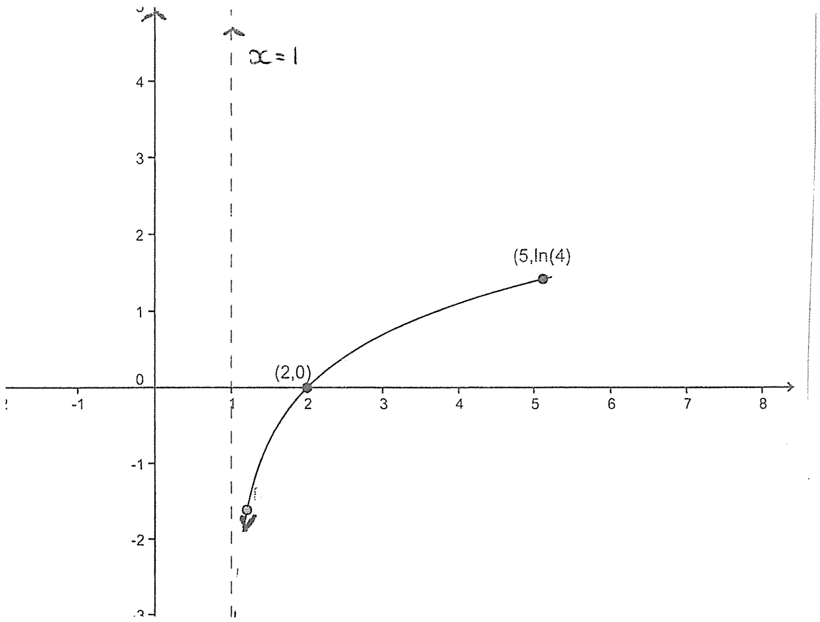
b) Find: $\int \cos^3 x \, . \sin x \quad dx$ **2**

c) (i) Prove $\tan(45° - A) = \dfrac{1 - \sin 2A}{\cos 2A}$ where A is acute. **3**

(ii) Hence, using a suitable value for A, find the exact value for $\tan 15°$ **2**

Question 6 – (12 marks) – (Start a new booklet) **Marks**

a) A sketch of $y = \ln(x-1)$ for $1 < x \leq 5$ is given.



(i) Show that the point $(e+1,\ 1)$ lies on the curve. 1

(ii) Calculate the exact area bound by the curve $y = \ln(x-1)$, the x-axis and the line $x = e+1$ 3

b) (i) Differentiate $x\log_e x - x$ 1

(ii) Hence, evaluate $\int_1^e \log_e x \ dx$ 3

c) If $y = e^x.\cos x$, find the value of k for which

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + ky = 0$$

is true for all x . 4

End of Paper

Question 1

YEAR 12 EXT 1 - MIDCOURSE HSC SOLNS

a) $\log_e 5 - \frac{1}{2}\log_e 16 = \log_e k$

LHS $\log_e 5 - \log_e 16^{\frac{1}{2}}$

$= \log_e 5 - \log_e 4$

$= \log_e \left(\frac{5}{4}\right)$

$\therefore k = \frac{5}{4}$

b) (i) $\frac{2\pi}{9}$ radians

π radians $= 180^\circ$

$\therefore \frac{2\pi}{9}$ radians $= 180^\circ \times \frac{2}{9}$

$= 40^\circ$

(ii) $\cos\left(\frac{-11\pi}{6}\right) + \sin\left(\frac{2\pi}{3}\right)$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$

$= \frac{2\sqrt{3}}{2}$

$= \sqrt{3}$

c) $y = \log_e\left(\frac{x+1}{\sqrt{x}}\right)$

$y = \log_e(x+1) - \log_e \sqrt{x}$

$y = \log_e(x+1) - \log_e x^{\frac{1}{2}}$

$y = \log_e(x+1) - \frac{1}{2}\log_e x$

then

$\frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{2} \times \frac{1}{x}$

$= \frac{1}{x+1} - \frac{1}{2x}$

$= \frac{2x - (x+1)}{2x(x+1)}$

$= \frac{2x - x - 1}{2x(x+1)}$

$= \frac{x-1}{2x(x+1)}$

d) $y = \log_e(x^2+2)$ where $x=1$.

$$\frac{dy}{dx} = \frac{2x}{x^2+2}$$

when $x=1$, $\frac{dy}{dx} = \frac{2}{1^2+2}$

$$\frac{dy}{dx} = \frac{2}{3}$$

$\therefore m = \frac{2}{3}$

For coordinates,

when $x=1$, $y = \log_e(1^2+2)$

$y = \log_e 3$

$(1, \log_e 3)$

So gradient of normal $= -\frac{3}{2}$

For equation, $x_1 = 1$, $y_1 = \log_e 3$

$y - y_1 = m(x - x_1)$

$y - \log_e 3 = -\frac{3}{2}(x-1)$

$y - \log_e 3 = -\frac{3}{2}x + \frac{3}{2}$

$2y - 2\log_e 3 = -3x + 3$

$3x + 2y = 3 + 2\log_e 3$

Qn 2

a) $\lim_{x \to 0} \frac{\sin\left(\frac{2x}{5}\right)}{3x}$

$= \frac{1}{3} \lim_{x\to 0} \frac{2}{5} \frac{\sin\left(\frac{2x}{5}\right)}{\left(\frac{2x}{5}\right)}$

$= \frac{2}{15} \lim_{x\to 0} \frac{\sin\left(\frac{2x}{5}\right)}{\left(\frac{2x}{5}\right)}$

$= \frac{2}{15}$

b) (i) Show that $\cos 4x = 2\cos^2 2x - 1$

LHS $\cos(2x+2x) = \cos 2x \cos 2x - \sin 2x \sin 2x$

$= \cos^2 2x - \sin^2 2x$

$= \cos^2 2x - (1 - \cos^2 2x)$

$= \cos^2 2x - 1 + \cos^2 2x$

$= 2\cos^2 2x - 1$

$=$ RHS

(ii) $\int \cos^2 2x \, dx = \int \frac{\cos 4x}{2} + \frac{1}{2} \, dx$

$= \frac{1}{2} \times \frac{1}{4} \sin 4x + \frac{x}{2} + C$

$= \frac{\sin 4x}{8} + \frac{x}{2} + C$

$\cos 4x = 2\cos^2 2x - 1$

$\frac{1}{2}\cos 4x = \cos^2 2x - 1$

$\therefore \frac{1}{2}\cos 4x + 1 = \cos^2 2x$

$\cos^2 2x = \frac{\cos 4x}{2} + 1$

c) $$\int_0^{\log_e 2} \frac{e^{3x}}{1+e^{3x}}\,dx$$

$$= \frac{1}{3}\int_0^{\log_e 2} \frac{3e^{3x}}{1+e^{3x}}\,dx$$

$$= \frac{1}{3}\left[\log_e(1+e^{3x})\right]_0^{\log_e 2}$$

$$= \frac{1}{3}\left[\log_e(1+e^{3\log_e 2}) - \log_e(1+e^0)\right]$$

$$= \frac{1}{3}\left[\log_e(1+e^{\log_e 2^3}) - \log_e 2\right]$$

$$= \frac{1}{3}\left[\log_e(1+2^3) - \log_e 2\right]$$

$$= \frac{1}{3}\left[\log_3 9 - \log_e 2\right]$$

$$= \frac{1}{3}\log_e\left(\frac{9}{2}\right)$$

d) (i) Area of Minor Sector $= 5\text{cm}^2$

$$\frac{1}{2}r^2\theta = 5$$

$$\frac{1}{2}\times 2^2\theta = 5$$

$$2\theta = 5$$

$$\therefore \theta = \frac{5}{2} \text{ radians.}$$

(ii) Length of major Arc AB.

$$\theta = 2\pi - \frac{5}{2}$$

$$\theta = \frac{4\pi}{2} - \frac{5}{2}$$

$$\theta = \frac{4\pi-5}{2}$$

$$l = r\theta.$$

$$l = 2\times\left(\frac{4\pi-5}{2}\right)$$

$$l = 4\pi - 5$$

$$l = 7.566370614$$

$$\therefore l = 7.6 \text{ cm. (to 1dec place).}$$

Qn3

a (i) $y = 2 + 2x$ $\qquad$ $y = \ln(3x+4)$

when $x = -1$,

$y = 2 + 2(-1)$
$y = 2 - 2$
$y = 0$.

$y = \ln(3(-1)+4)$
$y = \ln(-3+4)$
$y = \ln 1$
$y = 0$

$\therefore$ The line and the curve cross at the point $(-1, 0)$.

(ii) $\tan\theta = \left| \dfrac{m_1 - m_2}{1 + m_1 m_2} \right|$

For $y = 2 + 2x$

$\dfrac{dy}{dx} = 2$

Let $m_1 = 2$

For $y = \ln(3x+4)$

$\dfrac{dy}{dx} = \dfrac{3}{3x+4}$

when $x = -1$, $\dfrac{dy}{dx} = \dfrac{3}{-3+4}$

Let $m_2 = 3$.

Then $\tan\theta = \left| \dfrac{2-3}{1+2\times3} \right|$ $\qquad$ $\tan\theta = \left| \dfrac{-1}{7} \right|$

$\therefore \theta = 8°8'$

b) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^2 x \, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x - 1 \, dx$

$= \left[\tan x - x \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \left[\left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) \right]$

$= \sqrt{3} - \frac{\pi}{3} - \left(1 - \frac{\pi}{4}\right)$

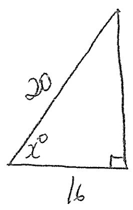
$= \sqrt{3} - \frac{\pi}{3} - 1 + \frac{\pi}{4}$

$= \sqrt{3} - 1 - \frac{\pi}{12}$

c)

Area of Minor Segment $= \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2} \times 20^2(\theta - \sin\theta)$



$= \frac{1}{2} \times 20^2(1.286\ldots - \sin 1.2868\ldots)$

$\doteqdot 65 \text{ cm}^2$

$\cancel{\tan}\ \cos x^\circ = \frac{16}{20}$

$x = 36^\circ 52'$

Now π radians $= 180^\circ$

$\frac{\pi}{180}$ radians $= 1^\circ$

$\frac{36.8\ldots\pi}{180}$ radians $= 36^\circ 52'$

We need $2x = 73^\circ 44'$

$\frac{73^\circ 44' \pi}{180}$ radians $= 73^\circ 44'$

1.286889435 radians $= 73^\circ 44'$

OR Let $P\hat{O}Q = 2\alpha$

$\cos\alpha = \frac{16}{20}$

$= \frac{4}{5}$

Area minor segment

$= \frac{1}{2}r^2(\theta - \sin\theta)$

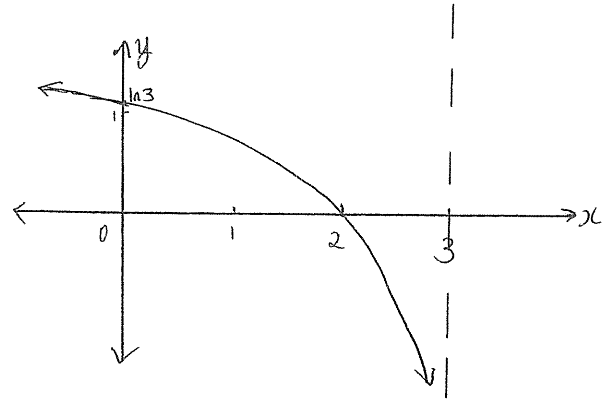
$= \frac{1}{2}r^2[2\alpha - \sin 2\alpha]$

$= 200\left[2\cos^{-1}\left(\frac{4}{5}\right) - 2.\frac{3}{5}.\frac{4}{5}\right]$

$= 65.400\ldots$

$= 65.4 \text{ cm}^2$ (correct to 1 dec pl)

d) $y = \ln(3-x)$



when $x = 0$,

$y = \ln 3$

Qn4

a (i) $\frac{d}{dx}\left[\frac{\log_e x}{x^2}\right]$

$$= \frac{x^2 \cdot \frac{1}{x} - \log_e x \cdot 2x}{(x^2)^2}$$

$$= \frac{x - 2x\log_e x}{x^4}$$

$$= \frac{1-2\log_e x}{x^3}$$

(ii) $\frac{dy}{dx} = \frac{1-2\log_e x}{x^3}$

When $x=e$,

$$\frac{dy}{dx} = \frac{1-2\log_e e}{e^3}$$

$$m = \frac{1-2}{e^3}$$

$$m = \frac{-1}{e^3}$$

For coordinates, when $x=e$,

$$y = \frac{\log_e e}{e^2}$$

$$y = \frac{1}{e^2}$$

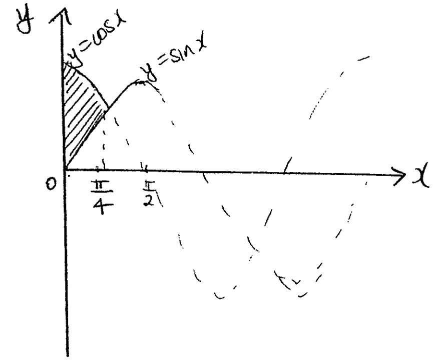
$\therefore \left(e, \frac{1}{e^2}\right)$

Eqn of tangent

$$y - \frac{1}{e^2} = -\frac{1}{e^3}(x-e)$$

$$e^3 y - e = -x + e$$

b) $y = \cos x$ $y = \sin x$



$$V = \pi \int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x \, dx$$

$$V = \pi \int_0^{\frac{\pi}{4}} \cos 2x \, dx$$

$$V = \pi \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$

$$V = \frac{\pi}{2}\left[\sin 2x\right]_0^{\frac{\pi}{4}}$$

$$V = \frac{\pi}{2}\left[\sin\frac{2\pi}{4} - \sin 0\right]$$

$$V = \frac{\pi}{2}\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$V = \frac{\pi}{2}[1]$$

$\therefore V = \frac{\pi}{2}$ units^3

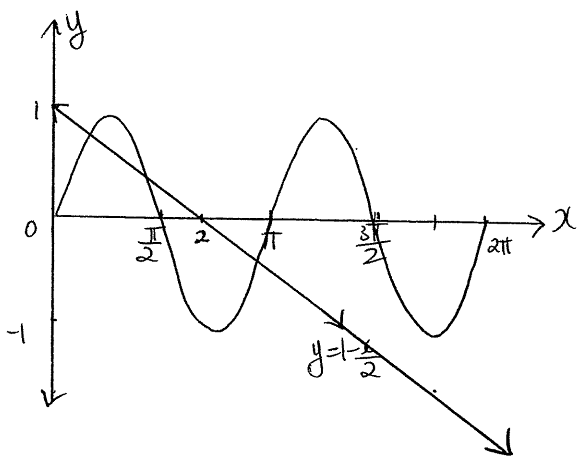
Qn5

a) (i) $y = \sin 2x$ for $0 \le x \le 2\pi$

$a = 1$, Period $= \frac{2\pi}{2}$

Period $= \pi$ units.

(ii)



$\frac{x}{2} + \sin 2x = 1$

$\sin 2x = 1 - \frac{x}{2}$

For $y = 1 - \frac{x}{2}$

When $x=0$, $y=1$

$y=0$, $x=2$

(iii) There are 3 solutions.

b) $\int \cos^3 x \sin x \, dx$

$= \int \sin x . (\cos x)^3 dx$

$= -\int -\sin x (\cos x)^3 dx$

$= -\left[\frac{(\cos x)^4}{4}\right] + C$

$= -\frac{\cos^4 x}{4} + C$.

c) (i) Prove $\tan(45° - A) = \frac{1 - \sin 2A}{\cos 2A}$ where A is acute

LHS. $\tan(45° - A) = \frac{\tan 45° - \tan A}{1 + \tan 45° \tan A}$

$= \frac{1 - \tan A}{1 + \tan A}$

$= \frac{1 - \frac{\sin A}{\cos A}}{1 + \frac{\sin A}{\cos A}}$ $\times \cos A$. / $\times \cos A$

$= \frac{\cos A - \sin A}{\cos A + \sin A}$

c) $y = e^{\sin x} - 1 \qquad 0 \le x \le 2\pi$

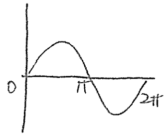
(i) x-intercepts when $y = 0$

$e^{\sin x} - 1 = 0$

$e^{\sin x} = 1$

$\ln e^{\sin x} = \ln 1$

$\sin x = 0$



$\therefore x = 0, \pi, 2\pi$

(ii) Possible stationary pts occur when $\frac{dy}{dx} = 0$. $\qquad \frac{dy}{dx} = \cos x e^{\sin x}$

ie $\cos x e^{\sin x} = 0$

ie $\cos x = 0$ OR $e^{\sin x} \neq 0$ no solns

$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$

OR Test $\frac{dy}{dx}$ either side.

To test nature.

$$\frac{d^2y}{dx^2} = e^{\sin x}.(-\sin x) + \cos x . \cos x e^{\sin x}$$

$$= -\sin x e^{\sin x} + \cos^2 x e^{\sin x}$$

$$\frac{d^2y}{dx^2} = e^{\sin x}(\cos^2 x - \sin x).$$

when $x = \frac{\pi}{2}$, $\quad \frac{d^2y}{dx^2} = e^{\sin\frac{\pi}{2}}(\cos^2\frac{\pi}{2} - \sin\frac{\pi}{2})$

$= e(0 - 1)$

$= -e$

$\frac{d^2y}{dx^2} < 0 \quad \therefore$ max when $x = \frac{\pi}{2}$.

For coordinates, when $x = \frac{\pi}{2}$, $\quad y = e^{\sin\frac{\pi}{2}} - 1$

$y = e - 1$

Local maximum at $(\frac{\pi}{2}, e-1)$

when $x = \frac{3\pi}{2}$, $\quad \frac{d^2y}{dx^2} = e^{\sin\frac{3\pi}{2}}(\cos^2\frac{3\pi}{2} - \sin\frac{3\pi}{2})$

$= e^{-1}(0 - -1)$

$= \frac{1}{e} > 0$

$\therefore$ minimum turning pt.

For coordinates, when $x = \frac{3\pi}{2}$, $\quad y = e^{\sin\frac{3\pi}{2}} - 1$

$y = e^{-1} - 1$

Local minimum at $(\frac{3\pi}{2}, \frac{1}{e} - 1)$

$$= \frac{\cos A - \sin A}{\cos A + \sin A} \times \frac{\cos A - \sin A}{\cos A - \sin A}$$

$$= \frac{\cos^2 A - 2\sin A\cos A + \sin^2 A}{\cos^2 A - \sin^2 A}$$

$$= \frac{1 - 2\sin A\cos A}{\cos^2 A - \sin^2 A}$$

Since
$\sin 2A = 2\sin A\cos A$
$\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{1 - \sin 2A}{\cos 2A}$$

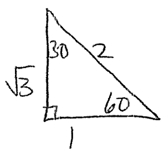
$= \underline{RHS}$

(ii) $\tan 15^\circ$

$= \tan(45^\circ - 30^\circ)$

$$= \frac{1 - \sin(2\times 30^\circ)}{\cos(2\times 30^\circ)}$$

$$= \frac{1 - \sin 60^\circ}{\cos 60^\circ}$$



$$= \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$= \frac{\frac{2-\sqrt{3}}{2}}{\frac{1}{2}} = \underline{2-\sqrt{3}} \text{ units.}$$

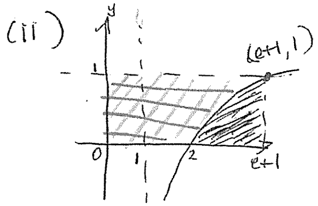
Question 6

a) (i) when $x = e+1$,

$y = \ln(e+1-1)$
$y = \ln e$
$y = 1$

$\therefore (e+1, 1)$ lies on the curve.

(ii)



Area = Area of rectangle $- \int_0^1 e^y + 1 \, dy$.

$= (e+1) - [e^y + y]_0^1$

$= e+1 - [(e+1) - (e^0 + 0)]$

$= e+1 - [e+1-1]$

$= \underline{1 \text{ unit}^2}$

$y = \ln(x-1)$
$x - 1 = e^y$
$\underline{x = e^y + 1}$

or: $\frac{d}{dx}(x-1)\ln(x-1) = \ln(x-1) + 1$

$A = \int_2^{e+1} \ln(x-1)\,dx = \int_2^{e+1} (\ln(x-1) + 1)\,dx - \int_2^{e+1} 1\,dx$

$= \left[(x-1)\ln(x-1) - x\right]_2^{e+1}$

$= (e\ln e - e - 1) - (\ln 1 - 2)$

$= \cancel{e - e} - 1 - \cancel{0} + 2 = 1\,u^2$

(b) (i) $x\log_e x - x$

Differentiating,

$\log_e x . 1 + x . \frac{1}{x} - 1$

$= \log_e x + 1 - 1$

$= \log_e x.$

$\therefore \frac{d}{dx}(x\log_e x - x) = \log_e x$

(ii) $\int_1^e \log_e x \, dx = \int_1^e \frac{d}{dx}(x\log_e x - x)\,dx$

$= \left[x\log_e x - x\right]_1^e$

$= \left[(e\log_e e - e) - (\log_e 1 - 1)\right]$

$= \left[(e-e) - (0-1)\right]$

$= \underset{\sim}{1}$

c) $y = e^x . \cos x$

$\frac{dy}{dx} = \cos x . e^x + e^x . -\sin x$

$= e^x(\cos x - \sin x)$

$\frac{d^2y}{dx^2} = (\cos x - \sin x).e^x + e^x(-\sin x - \cos x)$

$= e^x(\cos x - \sin x) - e^x(\sin x + \cos x)$

$= e^x \cancel{\cos x} - e^x\sin x - e^x \sin x - e^x \cancel{\cos x}$

$\frac{d^2y}{dx^2} = -2e^x \sin x$

Substituting,

$-2e^x\sin x - 2e^x(\cos x - \sin x) + ke^x\cos x = 0$

$-2e^x\cancel{\sin x} - 2\cos x e^x + 2e^x\cancel{\sin x} + ke^x\cos x = 0$

$-2\cos x e^x + ke^x \cos x = 0$

$-2e^x\cos x + ke^x\cos x = 0$

$e^x\cos x(-2+k) = 0$

$e^x\cos x(k-2) = 0$

$\therefore \underset{\sim}{k=2}$